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On behaviour of free-surface profiles for bounded steady water waves

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Abstract

The paper deals with the classical non-linear problem of steady two-dimensional waves on water of finite depth. The problem is formulated so that it describes all waves without *stagnation points* on the free-surface profiles that are *bounded* themselves and have *bounded* slopes. By virtue of reducing the problem to an integro-differential equation the following three results are proved. First, there are no waves when the flow is *critical*. Second, there are no waves having profiles totally above the upper boundary of the uniform *subcritical* stream. Finally, only two types of the free-surface behaviour are possible at positive (or/and negative) infinity: the profile either oscillates infinitely many times around the upper boundary of the subcritical uniform stream or asymptotes the upper level of a uniform stream (subcritical or supercritical). The latter assertion is proved under additional assumption that the slope of the free surface is a uniformly continuous function.

Résumé

Nous étudions le problème bidimensionnel, non linéaire d'ondes permanentes à la surface libre d'eaux d'une profondeur finie. Notre formulation du problème englobe toutes les ondes dont la surface libre satisfait deux conditions : leurs profils et leurs pentes sont *bornés* ; il n'existe pas de points de stagnation sur leurs profils. Pour le problème rédiut à une équation integro-differentielle, nous démontrerons trois résultats : Premièrement, il n'existe pas d'ondes dans le cas *critique*. Deuxièmement, il n'existe pas d'ondes quand des profils de la surface libre sont au-dessus de la frontière supérieure du courant uniformément, *souscritique*. Finalement, nous décrivons les deux types du comportement de la surface libre à l'infini (positif ou negatif ou bien les deux). Le profil ou bien oscille indéfiniment au voisinage de la frontière supérieure du courant souscritique ou bien tend vers la frontière supérieure du courant uniformément (souscritique ou surcritique). Cette dernière proposition est démontrée sous l'hypothèse supplémentaire que la pente de la surface libre est une fonction uniformément continue.

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1. Introduction

This paper concerns two-dimensional steady waves on water of finite depth. We consider the free-boundary problem that describes these waves under usual assumptions (gravity is taken into account, whereas the surface tension is neglected). Our statement of the problem covers all bounded waves (periodic, solitary, whatever) such that there are no stagnation points on the free-surface profiles and the latter have bounded slopes. For this problem, the following three results are proved.

First, we prove that steady waves cannot exist on the free surface of the critical flow. The absence of waves in the critical case is a commonplace for experts in fluid mechanics, but this fact was established rigorously only for two particular kinds of waves, namely, Stokes waves (periodic waves whose profiles rise and fall exactly once per period), and solitary waves (such a wave has a pulse-like profile that is symmetric about the vertical line through wave's crest and monotonically decreases away from it). These proofs were obtained by Keady and Norbury [11] for Stokes waves (see also Benjamin [5]), and by Amick and Toland [3] for solitary waves. There are other proofs for the solitary-wave case (see Keady and Pritchard [12], and McLeod [15]), but those proofs involve an extra assumption: Bernoulli's equation must hold throughout the water domain.

Our second assertion deals with the case of subcritical flow. It says that there are no steady waves that have their free-surface profiles totally above the upper flat boundary of the subcritical uniform stream. Again, this fact was earlier obtained only for Stokes and solitary waves in the papers [11] (see also [5]) and [3], respectively.

The third result is proved under additional assumption that the slope of the free surface is a uniformly continuous function. This assertion proves that only two types of behaviour are allowed for the free-surface profiles at positive (or/and negative) infinity. A profile either oscillates infinitely many times around the upper boundary of the subcritical uniform stream or asymptotes the upper level of one of the uniform streams (subcritical or supercritical). The first type of behaviour comprises, in particular, periodic waves. On the other hand, it is well known that solitary waves asymptote the upper level of the supercritical uniform stream, thus having the second type of behaviour. The existence of steady waves with profiles that asymptote the upper level of the subcritical uniform stream is an open question. However, it follows from our second result that free-surface profiles of the latter type cannot exist provided they are totally above the upper subcritical level. Moreover, it was proved in [13], Theorem 4.2, that non-trivial profiles lying totally below the same level are also impossible.

Before 1980, studies of steady water waves were focused on two kinds of them, namely, Stokes and solitary waves (see the survey article [10] by Groves). Chen and Saffman [8] were first who discovered numerically that periodic waves with two and three crests per period do exist on infinitely deep water. This breakthrough was extended by Aston [4], whose numerical results based on group-theoretic methods showed that there are waves having rather many crests per period. The existence of periodic waves on deep water that distinguish from Stokes ones was rigorously proved by Buffoni et al. [6]. In the case of finite depth, steady waves with more than one crest per period were found numerically by Craig and Nicholls [9].

All waves discovered in the papers [4,6,8,9], and some other works belong to the class of bounded waves described at the beginning of Introduction. For this class, we prove the results outlined above, thus continuing studies initiated in [13], where some bounds for wave characteristics were established. These bounds known earlier only for Stokes waves were extended in [13] to the whole class of waves that have bounded profiles and slopes. In that paper, it was also shown that there are no non-trivial waves with profiles below the subcritical level. Some facts about the asymptotic behaviour of bounded waves were obtained in [13] as well.

It should be mentioned that we treat only the case of water of finite depth here. However, assertions similar to the second and third our results might be also true for water of infinite depth. The authors are going to investigate the latter case in the future.

1.1. Statement of the problem

We consider steady gravity waves in a horizontal open channel of uniform rectangular cross-section occupied by an inviscid, incompressible, and heavy fluid, say water, bounded above by a free surface (the surface tension is neglected there), whereas a rigid bottom bounds water from below. The water motion is assumed to be two-dimensional and irrotational, and so there exists a velocity potential. In the problem's statement involving dimensional variables, the values of two parameters are supposed to be given: the volume rate of flow per unit span Q and the Bernoulli constant R (also referred to as the total head). The following inequality $R \geqslant R_c = \frac{3}{2}(Qg)^{2/3}$, where g is the acceleration due to gravity, is shown to be a necessary condition for the existence of steady waves (see Theorem 2.3(i) in [13]). If $R > R_c$, then one finds that $2(R - g\xi) = (Q/\xi)^2$ (Bernoulli's equation for the uniform stream of depth ξ) has two positive roots $\xi_- < \xi_c = (Q^2/g)^{1/3}$ and $\xi_+ > \xi_c$, which coincide with ξ_c when $R = R_c$; the values ξ_-, ξ_c, and ξ_+ are referred to as the depths of the *supercritical, critical*, and *subcritical* uniform streams, respectively.

Below, we use a non-dimensional statement of the problem obtained as follows. In order to obtain non-dimensional coordinates and the free-surface profile, we use ξ_+ as the length unit, whereas Q serves for the same purpose in the case of the velocity potential (see details in [13], where we applied ξ_- as the unit of length instead of ξ_+). In appropriate (non-dimensional) Cartesian coordinates (x, y), the bottom is given by $y = -1$, and gravity acts in the negative y-direction. The frame of reference is chosen so that the velocity field and the free-surface profile are time-independent. This profile is supposed to be the graph of an unknown C^1-function, say $y = \eta(x)$, and so the water domain is $D = \{x \in \mathbb{R},\ -1 < y < \eta(x)\}$, and the dimensionless velocity potential ϕ is given in $\bar{D}$. Thus we are going to consider:

Problem $\mathbf{P}_{(\phi,\eta)}$. For a given value of parameter $\lambda \in [1, +\infty)$ ($\lambda = g\xi_+^3/Q^2$) find a pair (ϕ, η) with the following properties:

- $\eta(x)$, $|\eta_x(x)|$ are bounded for all $x \in \mathbb{R}$, and
$$\inf_{x\in\mathbb{R}} \eta(x) > -1, \qquad \sup_{x\in\mathbb{R}} \eta(x) < \frac{1}{2\lambda}; \tag{1}$$
- $\phi \in C^1(\bar{D}) \cap C^2(D)$ satisfies the boundary value problem:
$$\phi_{xx} + \phi_{yy} = 0, \quad (x, y) \in D; \tag{2}$$
$$\phi_y = 0, \qquad y = -1, \quad x \in \mathbb{R}; \tag{3}$$
$$\phi_y = \eta_x\phi_x, \qquad y = \eta(x), \quad x \in \mathbb{R}; \tag{4}$$
$$|\nabla\phi|^2 + 2\lambda\eta = 1, \qquad y = \eta(x), \quad x \in \mathbb{R}; \tag{5}$$
- the following relation holds:
$$\int_{-1}^{\eta(x)} \phi_x(x, y)\,\mathrm{d}y = 1. \tag{6}$$

Condition (5) implies the non-strict inequality $\eta(x) \leqslant 1/(2\lambda)$ for all $x \in \mathbb{R}$. However, there is an example (see e.g. Amick et al. [2]), in which the fact that $\eta \in C^1$ is violated at the point, where equality holds. Therefore, we impose the second inequality (1), and so $|\nabla\phi(x, \eta(x))|$ does not vanish for all $x \in \mathbb{R}$ (there are no stagnation points on the free surface). By Lewy's theorem [14], the latter condition yields that $y = \eta(x)$ is a real-analytic curve, across which one can extend the velocity potential ϕ harmonically.

1.2. Formulation of the results

Here we formulate rigorously our results described above in hydrodynamic terms, but first we note that problem $\mathrm{P}_{(\phi,\eta)}$ defines ϕ only up to an additive constant. Taking this into account, a direct calculation based on Eqs. (5) and (6) shows that for $\lambda \geqslant 1$ the problem has two *trivial* solutions:
$$(x + b, 0) \quad \text{and} \quad \big([1 + \eta_*]^{-1}x + b, \eta_*\big), \tag{7}$$
where b is an arbitrary constant, and
$$\eta_* = \frac{1 - 4\lambda + \sqrt{1 + 8\lambda}}{4\lambda} \leqslant 0.$$
The first (second) of these solutions describes the *subcritical* (*supercritical*, respectively) uniform stream. Moreover, these solutions coincide in the *critical* case $\lambda = 1$. Solutions other than those given by formulae (7) will be referred to as *non-trivial*.

Theorem 1. (i) *If* $\lambda = 1$, *then problem* $\mathrm{P}_{(\phi,\eta)}$ *has only a trivial solution* $(x+b, 0)$.

(ii) *Let problem* $\mathrm{P}_{(\phi,\eta)}$ *with* $\lambda > 1$ *in condition* (5) *have a solution such that*

$$\eta(x) \geqslant 0 \quad \text{for all } x \in \mathbb{R}. \tag{8}$$

Then $(\phi, \eta) = (x + b,\, 0)$.

Theorem 2. *Let problem* $\mathrm{P}_{(\phi,\eta)}$ *with* $\lambda > 1$ *in condition* (5) *have a solution such that* η_x *is uniformly continuous on* $\mathbb{R}$. *Then there are only two types of behaviour for* η *at the positive* (*or/and negative*) *infinity*:

(I) η *changes its sign infinitely many times*;
(II) $\eta(x)$ *has a limit as* x *tends to infinity and this limit is equal either to* 0 *or to* η_*.

What follows is the proof of these theorems divided into three sections. In Section 2, problem $\mathrm{P}_{(\phi,\eta)}$ is reduced to an integro-differential equation, which is of interest on its own. An auxiliary assertion is proved in Section 3, whereas Section 4 contains the main body of proofs of Theorems 1 and 2.

2. Integro-differential equation

Our proofs are based on an integro-differential equation for η, to which problem $\mathrm{P}_{(\phi,\eta)}$ reduces. In this section, we give a detailed derivation of this equation. The first step is to introduce a *stream function* ψ (a harmonic function conjugate to ϕ in D). The appropriate choice of an additive constant for ψ leads to the following boundary value problem equivalent to (2)–(6):

$$\psi_{xx} + \psi_{yy} = 0, \quad (x, y) \in D; \tag{9}$$

$$\psi = 0, \qquad y = -1, \quad x \in \mathbb{R}; \tag{10}$$

$$\psi = 1, \qquad y = \eta(x), \quad x \in \mathbb{R}; \tag{11}$$

$$|\nabla\psi|^2 + 2\lambda\eta = 1, \qquad y = \eta(x), \quad x \in \mathbb{R}. \tag{12}$$

It is well known that $x + iy \mapsto \phi + i\psi$ is a conformal mapping of D onto $\mathbb{R} \times (0, 1)$. The next step is to use the hodograph transform, that is, to consider $y(\phi, \psi)$ (the imaginary part of the inverse conformal mapping), as the new unknown function that satisfies the following boundary value problem (every relation of which has the corresponding counterpart in problem (9)–(12)):

$$y_{\phi\phi} + y_{\psi\psi} = 0, \quad (\phi, \psi) \in \mathbb{R} \times (0, 1); \tag{13}$$

$$y = -1, \qquad \psi = 0, \quad \phi \in \mathbb{R}; \tag{14}$$

$$y = \eta, \qquad \psi = 1, \quad \phi \in \mathbb{R}; \tag{15}$$

$$\left(y_\phi^2 + y_\psi^2\right)^{-1} + 2\lambda y = 1, \qquad \psi = 1, \quad \phi \in \mathbb{R}. \tag{16}$$

Note that this problem allows us to deduce the Hölder continuity of $x(\phi, \psi)$ in the following way. Since $|y| \leqslant 1 + \|\eta\|_{L^\infty(\mathbb{R})}$, the local L^p-estimate for problem (13)–(15) (see e.g. [1, Chapter 5]) yields that

$$\left(\int_0^1 \int_t^{t+1} |\nabla y|^p \,\mathrm{d}\phi \,\mathrm{d}\psi\right)^{1/p} \leqslant C_p\Big[1 + \sup_{\phi\in\mathbb{R}}\big(\big|\eta(\phi)\big| + \big|\eta_\phi(\phi)\big|\big)\Big], \quad p \in (1, \infty).$$

Here and below we write $\eta(\phi)$ instead of $\eta(x(\phi, 1))$ and hope that this will not cause a confusion. Since x and y are harmonic conjugate, the same estimate holds with y changed to x. Then an appropriate embedding theorem yields that

$$\big|x(\phi_1, \psi_1) - x(\phi_2, \psi_2)\big| \leqslant C\big(|\phi_1 - \phi_2| + |\psi_1 - \psi_2|\big)^{\frac{p-1}{p}}, \tag{17}$$

when $|\phi_1 - \phi_2| + |\psi_1 - \psi_2| \leqslant 1$, where the constant C depends only on p, $\|\eta\|_{L^\infty(\mathbb{R})}$, and $\|\eta_\phi\|_{L^\infty(\mathbb{R})}$.

Now we continue the derivation of our integro-differential equation. Conditions (15) and (16) imply the following relation:

$$y_\psi(\phi, 1) = \left[\frac{1}{1 - 2\lambda\eta(\phi)} - \eta_\phi^2(\phi)\right]^{1/2}. \tag{18}$$

On the other hand, let us consider the Fourier transform:

$$(Fy)(\tau, \psi) = \int_{-\infty}^{+\infty} y(\phi, \psi) \mathrm{e}^{i\tau\phi} \, \mathrm{d}\phi$$

(it must be understood in the sense of the distribution space $\mathcal{S}'$). Then we get from (13) and (14) that

$$\big[F(y+1)\big](\tau, \psi) = A(\tau) \sinh(\tau\psi).$$

Differentiating this and eliminating A, we obtain:

$$\big[F(y+1)\big](\tau, \psi) = \tau^{-1} \tanh(\tau\psi)\big[F(y_\psi)\big](\tau, \psi).$$

Putting $\psi = 1$ in this equality and applying the inverse Fourier transform, we obtain the following convolution equation (it coincides with that of Byatt-Smith [7] up to non-dimensional):

$$\eta(\phi) + 1 = \int_{-\infty}^{+\infty} k(\phi - \varphi)\left[\frac{1}{1 - 2\lambda\eta(\varphi)} - \eta_\varphi^2(\varphi)\right]^{1/2} \mathrm{d}\varphi.$$

Here relation (18) is used, whereas the kernel is the following inverse Fourier transform:

$$k(\phi) = \big[F^{-1}\big(\tau^{-1} \tanh\tau\big)\big](\phi).$$

Since $\int_{-\infty}^{+\infty} k(\phi)\,\mathrm{d}\phi = 1$, we get another form of the convolution equation,

$$\eta = B\left[\frac{L(\eta, \eta_\phi)}{S(\eta)} - 1\right], \tag{19}$$

where B stands for the convolution operator with the kernel k and the following notation is used:

$$S(\eta) = \sqrt{1 - 2\lambda\eta}, \qquad L(\eta, \eta_\phi) = \sqrt{1 - \eta_\phi^2 S^2}.$$

Since B is invertible (its symbol $\tau^{-1}\tanh\tau$ does not vanish), Eq. (19) can be written as follows:

$$\big(B^{-1} - \lambda I\big)\eta = \eta^2 H_0(\eta) - \eta_\phi^2 H_1(\eta, \eta_\phi), \tag{20}$$

where I is the identity operator, and

$$H_0 = \frac{2\lambda^2(2 + S)}{S(1 + S)^2}, \qquad H_1 = \frac{S}{1 + L}. \tag{21}$$

Indeed, after some algebra one finds that

$$\frac{2\lambda\eta}{S(1 + S)} - \lambda\eta = \eta^2 H_0 \quad \text{and} \quad \frac{L}{S} - 1 - \frac{2\lambda\eta}{S(1 + S)} = -\eta_\phi^2 H_1.$$

In order to write Eq. (20) in the form appropriate for proving our theorems, we note that the symbol of $B^{-1} - \lambda I$ is equal to $\tau\coth\tau - \lambda$. Being an even function, this symbol has two zeroes when $\lambda > 1$, say $\pm\lambda_*$, $\lambda_* > 0$, whereas $\lambda_* = 0$ is the double zero of the symbol when $\lambda = 1$. Hence we can factorise the latter operator as follows:

$$B^{-1} - \lambda I = -Q_\lambda\big(\mathrm{d}^2 + \lambda_*^2 I\big),$$

where the symbol of Q_λ is equal to $(\tau\coth\tau - \lambda)/(\tau^2 - \lambda_*^2)$, and so Q_λ is the convolution operator with the following kernel:

$$q_\lambda(\phi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\tau\coth\tau - \lambda}{\tau^2 - \lambda_*^2} \mathrm{e}^{-i\phi\tau} \, \mathrm{d}\tau. \tag{22}$$

Now Eq. (20) takes the form:

$$Q_\lambda\big(\mathrm{d}_\varphi^2+\lambda_*^2 I\big)\eta=\eta_\phi^2\,H_1(\eta,\eta_\phi)-\eta^2\,H_0(\eta). \tag{23}$$

It must be understood in the distribution sense because the assumptions of problem $\mathrm{P}_{(\phi,\eta)}$ guarantee that only the first derivative,

$$\eta_\phi=\frac{\eta_x}{\sqrt{(1-2\lambda\eta)(1+\eta_x^2)}}, \tag{24}$$

is bounded (formula (24) is a consequence of (4) and (5)). However, if we write Eq. (23) in the following equivalent form:

$$\big(\mathrm{d}_\phi Q_\lambda \mathrm{d}_\varphi+\lambda_*^2 Q_\lambda\big)\eta=\eta_\phi^2 H_1(\eta,\eta_\phi)-\eta^2 H_0(\eta), \tag{25}$$

then it has the ordinary interpretation. The latter fact follows from formula (29) (see below), according to which $\mathrm{d}_\phi Q_\lambda$ is a singular integral operator, whereas η_φ is bounded.

Some properties of q_λ will be required in what follows. From formula (22), we see that q_λ is an even function, and applying the residue theorem to the integral (22), we obtain that

$$q_\lambda(\phi)=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{n\mathrm{e}^{-n\pi|\phi|}}{n^2+(\lambda/\pi)^2}>0,\quad \phi\neq 0. \tag{26}$$

Therefore, we have,

$$q_\lambda''(\phi)=\pi\sum_{n=1}^{\infty}\frac{n^3\mathrm{e}^{-n\pi|\phi|}}{n^2+(\lambda/\pi)^2}\geqslant\frac{\pi^3\mathrm{e}^{-\pi|\phi|}}{\pi^2+\lambda^2},\quad \phi\neq 0. \tag{27}$$

Moreover, the following inequalities:

$$q_\lambda(\phi)\leqslant\frac{2}{\pi}\mathrm{e}^{-\pi|\phi|},\qquad \big|q_\lambda'(\phi)\big|\leqslant 2\mathrm{e}^{-\pi|\phi|},\qquad q_\lambda''(\phi)\leqslant 4\pi\mathrm{e}^{-\pi|\phi|} \tag{28}$$

hold for $|\phi|\geqslant\frac{1}{2}$.

Finally, let us note that

$$(Fq_\lambda)(\tau)\sim\frac{1}{2\pi}\left(\frac{1}{|\tau|}-\frac{\lambda}{\tau^2}\right)\sum_{k=0}^{\infty}\left(\frac{\lambda_*}{\tau}\right)^{2k}\quad\text{as } |\tau|\to\infty.$$

Hence the following asymptotic formula:

$$q_\lambda(\phi)=\frac{1}{\pi}\log\frac{1}{|\phi|}+r_\lambda\big(|\phi|\big),\quad \text{is valid as } |\phi|\to 0. \tag{29}$$

Here $r_\lambda\in C^\infty[0,1]$.

3. Lemma on wave profiles non-negative at infinity

Lemma proved in this section will be used in theorems' proofs, but it is of interest on its own.

Lemma. *Let $\lambda\geqslant 1$ and let η be a solution of Eq. (25). If there exists ϕ° such that $\eta(\phi)\geqslant 0$ for $\phi\geqslant\phi^\circ$, then*

$$\eta(\phi)\to\eta^\infty\quad \text{as } \phi\to+\infty, \tag{30}$$

where $\eta^\infty=\liminf_{\phi\to+\infty}\eta(\phi)\geqslant 0$.

Moreover, if η_x is uniformly continuous on $\mathbb{R}$, then

$$\eta(\phi),\ \eta_\phi(\phi)\to 0\quad \text{as } \phi\to+\infty.$$

Proof. For proving (30) by contradiction we suppose that there exist $h > 0$ and a sequence $\{\phi_n\}_1^\infty$ such that

$$\phi_n \to +\infty \quad \text{as } n \to \infty, \text{ but} \qquad \eta(\phi_n) \geqslant \eta^\infty + h \quad \text{for all } n. \tag{31}$$

The following two options must be considered:

(I) There exists $\phi_\ell \geqslant \phi^\circ$ such that $\eta(\phi) \geqslant \eta^\infty$ for $\phi \geqslant \phi_\ell$.
(II) There exists a sequence $\{\varphi_n\}_1^\infty$ such that

$$\varphi_n \to +\infty, \qquad \eta(\varphi_n) < \eta^\infty, \quad \text{and} \quad \eta(\varphi_n) \to \eta^\infty \quad \text{as } n \to \infty.$$

Case (I). First we assume that there exists a sequence $\phi_n^* \to +\infty$ such that $\eta(\phi_n^*) = \eta^\infty$ for $n = 1, 2, \ldots$; without loss of generality we take $\phi_n^* > \phi_n$ for all n. Then $\eta_\phi(\phi_n^*) = 0$, and so Eq. (25) reduces to

$$\big[\big(\mathrm{d}_\phi Q_\lambda \mathrm{d}_\varphi + \lambda_*^2 Q_\lambda\big)\eta\big](\phi_n^*) = -\eta^2(\phi_n^*)\, H_0\big(\eta(\phi_n^*)\big). \tag{32}$$

It is clear that

$$\big[(\mathrm{d}_\phi Q_\lambda \mathrm{d}_\varphi)\eta\big](\phi_n^*) = \big[(\mathrm{d}_\phi Q_\lambda \mathrm{d}_\varphi)\big(\eta - \eta^\infty\big)\big](\phi_n^*) = \int_{-\infty}^{+\infty} q_\lambda''(\phi_n^* - \varphi)\,\big[\eta(\varphi) - \eta^\infty\big]\,\mathrm{d}\varphi, \tag{33}$$

where the last integral is absolutely convergent since,

$$\eta(\varphi) - \eta^\infty = O\big((\varphi - \phi_n^*)^2\big) \quad \text{and} \quad q_\lambda(\phi_n^* - \varphi) = O\big((\varphi - \phi_n^*)^{-2}\big) \quad \text{as } \varphi \to \phi_n^*,$$

(see formula (29)).

Let us write the left-hand side in (32) as follows:

$$\left(\int_{-\infty}^{\phi_\ell} + \int_{\phi_\ell}^{+\infty}\right)\big\{q_\lambda''(\phi_n^* - \varphi)\big[\eta(\varphi) - \eta^\infty\big] + \lambda_*^2 q_\lambda(\phi_n^* - \varphi)\eta(\varphi)\big\}\,\mathrm{d}\varphi = I_- + I_+.$$

Taking into account formulae (26) and (27), and the definition of ϕ_ℓ, we see that the integrand of I_+ is non-negative. Since $|\eta_\phi|$ is bounded, relation (31) implies that there exists $d > 0$ depending only on h and $\|\eta_\phi\|_{L^\infty(\mathbb{R})}$ and such that

$$\eta(\phi) \geqslant \eta^\infty + \frac{h}{2} \quad \text{for } \phi \in [\phi_n - d, \phi_n + d].$$

Since $\eta(\phi_n^*) = \eta^\infty$, we also have that $\phi_n + d - \phi_n^*$ is greater than a positive constant that depend only on h and $\|\eta_\phi\|_{L^\infty(\mathbb{R})}$. Now using formula (27), we conclude that

$$I_+ \geqslant C_1 \int_{\phi_n - d}^{\phi_n + d} \mathrm{e}^{-\pi|\varphi - \phi_n^*|}\,\mathrm{d}\varphi \geqslant C_2 \mathrm{e}^{-\pi(\phi_n^* - \phi_n)}, \tag{34}$$

where C_1 and C_2 are positive constants depending only on h and $\|\eta_\phi\|_{L^\infty(\mathbb{R})}$.

Let us turn to estimating $|I_-|$ from above. Applying the first and third formulae (28), we get:

$$|I_-| \leqslant C \int_{-\infty}^{\phi_\ell} \mathrm{e}^{-\pi(\phi_n^* - \varphi)}\,\mathrm{d}\varphi = \frac{C}{\pi}\,\mathrm{e}^{-\pi(\phi_n^* - \phi_\ell)}, \tag{35}$$

where C is a positive constant depending only on $\|\eta_\phi\|_{L^\infty(\mathbb{R})}$. It follows from equality (32) that $I_+ \leqslant |I_-|$, and so inequalities (34) and (35) give us that $C_2 \mathrm{e}^{\pi\phi_n} \leqslant \frac{C}{\pi}\mathrm{e}^{\pi\phi_\ell}$, but this inequality cannot be true for large n. Therefore, the assumption made at the beginning of Case (I) does not hold. Consequently, there exists ϕ^+ such that $\eta(\phi) > \eta^\infty$ for $\phi \geqslant \phi^+$.

Let us show that for every $\delta > 0$ and every $N > 0$ there exists $\phi_0 > N$ with the following properties:

(i) η attains a local minimum at ϕ_0 such that $\eta(\phi_0) - \eta^\infty = \delta_0 \leqslant \delta$;

(ii) $\eta(\phi) - \eta^\infty > \delta_0$ for $\phi \in (\phi^+, \phi_0)$;
(iii) there exists $\phi_+ \in (\phi_0, +\infty)$ such that $\eta(\phi_+) - \eta^\infty = \delta_0$, $\eta(\phi) - \eta^\infty \geqslant \delta_0$ for $\phi \in [\phi_0, \phi_+]$, and for some $\phi_* \in (\phi_0, \phi_+)$ one has $\eta(\phi_*) \geqslant \eta^\infty + h$.

In order to find ϕ_0, ϕ_+, and ϕ_* we proceed as follows. Let $\Phi_0 > N$ be a number at which η attains local minimum such that $\eta(\Phi_0) - \eta^\infty \leqslant \delta$ and also $\eta(\phi) - \eta^\infty > \delta_0$ for $\phi \in (\phi^+, \Phi_0)$. (The existence of Φ_0 is guaranteed by the definition of η^∞ and (31).) It is clear that (i) and (ii) are true for this Φ_0. Let us check whether (iii) holds for the chosen Φ_0.

In view of the definition of η^∞, there exists the smallest $\Phi_+ > \Phi_0$ such that $\eta(\Phi_+) = \eta(\Phi_0)$. Then ϕ_* such that $\eta(\phi_*) \geqslant \eta^\infty + h$ either exists in the interval (Φ_0, Φ_+) or not. In the first case, we put ϕ_0 and ϕ_+ to be equal to Φ_0 and Φ_+, respectively. Otherwise, we change the previous value of Φ_0 to the smallest number in $[\Phi_+, +\infty)$ at which η attains its next local minimum (it is clear that the latter minimum is less than or equal to the previous one). The same considerations as above either provide the required ϕ_*, and hence, ϕ_0 and ϕ_+ as well, or we have again to consider the next local minimum. It is clear that after a finite number of iterations the three points ϕ_0, ϕ_+, and $\phi_* \in (\phi_0, \phi_+)$ that satisfy conditions (i)–(iii) must be found.

Now we put $N = \phi_n$, where n is large, and consider Eq. (25) for $\phi = \phi_0$, in which case this equation has the form (32) with ϕ_n^* changed to ϕ_0. Furthermore, we have that (cf. relation (33)):

$$\big[(\mathrm{d}_\phi Q_\lambda \,\mathrm{d}_\varphi)\eta\big](\phi_0) = \int_{-\infty}^{+\infty} q_\lambda''(\phi_0 - \varphi)\big[\eta(\varphi) - \eta(\phi_0)\big]\,\mathrm{d}\varphi.$$

Thus we obtain from (32) that

$$\left(\int_{-\infty}^{\phi^+} + \int_{\phi^+}^{\phi_+} + \int_{\phi_+}^{+\infty}\right)\big\{q_\lambda''(\phi_0 - \varphi)\big[\eta(\varphi) - \eta(\phi_0)\big] + \lambda_*^2 q_\lambda(\phi_0 - \varphi)\eta(\varphi)\big\}\,\mathrm{d}\varphi = I^{(-)} + I^{(0)} + I^{(+)} \leqslant 0.$$

This yields

$$I^{(0)} \leqslant \big|I^{(-)}\big| + \delta_0 \int_{\phi_+}^{+\infty} q_\lambda''(\phi_0 - \varphi)\,\mathrm{d}\varphi, \tag{36}$$

where for estimating $I^{(+)}$ we used (26), (27), and the following two facts: (1) for $\phi \in (\phi_+, +\infty)$ we have $\eta(\phi) \geqslant 0$; (2) the negative part of $\eta(\phi) - \eta(\phi_0)$ is less than or equal to δ_0.

In order to estimate $I^{(0)}$ from below, we note that the integrand of $I^{(0)}$ is non-negative and considerations similar to those used above for estimating I_+ can be applied. In particular, there exists $d = d(h, \|\eta_\phi\|_{L^\infty(\mathbb{R})}) > 0$ such that

$$\eta(\phi) - \eta(\phi_0) \geqslant \frac{h}{2} \quad \text{for } \phi \in [\phi_n - d, \phi_n + d] \cup [\phi_* - d, \phi_* + d].$$

Moreover, since $\eta(\phi) - \eta(\phi_0)$ vanishes at $\phi = \phi_0$, both numbers $\phi_0 - \phi_n - d$ and $\phi_* - d - \phi_0$ are greater than a certain positive constant depending only on h and $\|\eta_\phi\|_{L^\infty(\mathbb{R})}$. Then using formula (27), we get:

$$I^{(0)} \geqslant C_1\left(\int_{\phi_n - d}^{\phi_n + d} + \int_{\phi_* - d}^{\phi_* + d}\right)\mathrm{e}^{-\pi|\varphi - \phi_0|}\,\mathrm{d}\varphi \geqslant C_2\big[\mathrm{e}^{-\pi(\phi_0 - \phi_n)} + \mathrm{e}^{-\pi(\phi_* - \phi_0)}\big], \tag{37}$$

where C_1 and C_2 are positive constants depending only on h and $\|\eta_\phi\|_{L^\infty(\mathbb{R})}$.

For estimating $|I^{(-)}|$ from above we apply the first and third formulae (28), thus obtaining that $|I^{(-)}| \leqslant C\mathrm{e}^{-\pi(\phi_0 - \phi^+)}$ (cf. relation (35)). Combining this inequality, (36), and (37), and using again the third formula (28), we arrive at the following inequality:

$$C_2\big[\mathrm{e}^{-\pi(\phi_0 - \phi_n)} + \mathrm{e}^{-\pi(\phi_* - \phi_0)}\big] \leqslant C\big[\mathrm{e}^{-\pi(\phi_0 - \phi^+)} + \delta_0 \mathrm{e}^{-\pi(\phi_+ - \phi_0)}\big],$$

which leads to a contradiction. Indeed, we have that $C_2\,\mathrm{e}^{\pi\phi_n} > C\,\mathrm{e}^{\pi\phi^+}$ when n is large, whereas $C_2\,\mathrm{e}^{-\pi\phi_*} > C\,\delta_0\,\mathrm{e}^{-\pi\phi_+}$ because $\phi_* < \phi_+$ and $\delta_0 < \delta$, but δ can be chosen arbitrary small. This contradiction proves formula (30) under assumptions of Case (I) because $h > 0$ and a sequence with properties (31) does not exist in this case.

Case (II). Without loss of generality we can assume that

$$\eta(\varphi_n) = \min_{\phi\in[\phi_n,+\infty)} \eta(\phi) \quad \text{for } n = 1, 2, \ldots.$$

For every sufficiently small $\delta > 0$ we denote by ϕ_δ the rightmost root of the equation $\eta(\phi) = \eta^\infty - \delta$ (note that $\phi_\delta \geqslant \phi^\circ$ for small δ).

Let $\phi = \varphi_n$ and let n be sufficiently large, then Eq. (25) takes the form (32) with ϕ_n^* changed to φ_n. Furthermore, we have (cf. formula (33)):

$$\big[(\mathrm{d}_\phi Q_\lambda \mathrm{d}_\varphi)\eta\big](\varphi_n) = \int_{-\infty}^{+\infty} q_\lambda''(\varphi_n - \varphi)\big[\eta(\varphi) - \eta(\varphi_n)\big]\,\mathrm{d}\varphi,$$

and so we get from the new form of Eq. (32) that

$$\left(\int_{-\infty}^{\phi_\delta} + \int_{\phi_\delta}^{+\infty}\right)\big\{q_\lambda''(\varphi_n - \varphi)\big[\eta(\varphi) - \eta(\varphi_n)\big] + \lambda_*^2 q_\lambda(\varphi_n - \varphi)\eta(\varphi)\big\}\,\mathrm{d}\varphi = I_\delta^{(-)} + I_\delta^{(+)} \leqslant 0. \tag{38}$$

This yields $I_\delta^{(+)} \leqslant |I_\delta^{(-)}|$.

Let us estimate $I_\delta^{(+)}$ from below, for which purpose we note that the second term in braces in (38) is non-negative according to formula (26) and the definition of ϕ_δ. Hence we have:

$$I_\delta^{(+)} \geqslant \int_{\phi_\delta}^{+\infty} q_\lambda''(\varphi_n - \varphi)\big[\eta(\varphi) - \eta(\varphi_n)\big]\,\mathrm{d}\varphi.$$

Furthermore,

$$\eta(\varphi) - \eta(\varphi_n) \geqslant -\delta \quad \text{for } \varphi \in [\phi_\delta, +\infty),$$

whereas

$$\eta(\varphi) - \eta(\varphi_n) \geqslant 0 \quad \text{for } \varphi \in [\phi_n, +\infty).$$

Since $\eta(\phi_n) \geqslant \eta^\infty + h$, there exists $d = d(h, \|\eta_\phi\|_{L^\infty(\mathbb{R})}) > 0$ such that

$$\eta(\varphi) - \eta(\varphi_n) \geqslant \frac{h}{2} \quad \text{for } \varphi \in [\phi_n - d, \phi_n + d].$$

Moreover, $\varphi_n - \phi_n - d$ is greater than a certain positive constant depending only on h and $\|\eta_\phi\|_{L^\infty(\mathbb{R})}$ because $\eta(\varphi_n) < \eta^\infty$. Now using formula (27) and the third formula (28), we obtain that

$$I_\delta^{(+)} \geqslant C_1 \int_{\phi_n-d}^{\phi_n+d} \mathrm{e}^{-\pi(\varphi_n-\varphi)}\,\mathrm{d}\varphi - C_2\delta \int_{\phi_\delta}^{\phi_n-d} \mathrm{e}^{-\pi(\varphi_n-\varphi)}\,\mathrm{d}\varphi \geqslant (C_3 - C_4\delta)\mathrm{e}^{-\pi(\varphi_n-\phi_n)}, \tag{39}$$

where $C_1, \ldots, C_4$ are positive constants depending only on h and $\|\eta_\phi\|_{L^\infty(\mathbb{R})}$.

For estimating $|I_\delta^{(-)}|$ from above we apply the first and third formulae (28), thus obtaining $|I_\delta^{(-)}| \leqslant C\mathrm{e}^{-\pi(\varphi_n-\phi_\delta)}$ (cf. inequality (35)). Combining the last inequality and (39), we arrive at the following inequality:

$$(C_3 - C_4\,\delta)\mathrm{e}^{\pi\phi_n} \leqslant C\mathrm{e}^{\pi\phi_\delta},$$

which leads to a contradiction. Indeed, we have an arbitrarily large positive value in the left-hand side; for this purpose we first take sufficiently small δ (in order to make positive the expression in brackets), thus also fixing the right-hand side, and then take n to be large.

Now we conclude that formula (30) holds in both Cases (I) and (II).

The next step is to prove that

$$\eta_\phi(\phi) \to 0 \quad \text{as } \phi \to +\infty, \tag{40}$$

for which purpose we need the assumption that η_x is uniformly continuous on $\mathbb{R}$. Combining formula (24) and the fact that the mapping $(\phi, \psi) \mapsto (x, y)$ is Hölder continuous (see formula (17)), we conclude that η_ϕ is also uniformly continuous with respect to its argument.

Let us assume contrary to (40) that there exist $h > 0$ and a sequence $\{\varphi_n^*\}_1^\infty$ such that

$$\varphi_n^* \to +\infty \quad \text{as } n \to +\infty, \text{ but} \qquad \eta_\phi(\varphi_n^*) \geqslant h \quad \text{for all } n. \tag{41}$$

Since η_ϕ is uniformly continuous, one can find δ such that

$$\left|\eta_\phi(\phi_1) - \eta_\phi(\phi_2)\right| < h/2 \quad \text{provided } |\phi_1 - \phi_2| < \delta. \tag{42}$$

For some $\varphi_n^\circ \in [\varphi_n^*, \varphi_n^* + \delta]$ we have $\eta(\varphi_n^* + \delta) - \eta(\varphi_n^*) = \eta_\phi(\varphi_n^\circ)\,\delta$, and so formulae (41) and (42) yield that $\eta(\varphi_n^* + \delta) - \eta(\varphi_n^*) \geqslant h\delta/2$, which contradicts relation (30). Hence $h > 0$ and a sequence with properties (41) cannot exist.

It remains to consider the case when instead of (41) we have that

$$\eta_\phi(\varphi_n^*) \leqslant -h \quad \text{and} \quad \varphi_n^* \to +\infty \quad \text{as } n \to +\infty.$$

However, this leads to a contradiction with relation (30) in the same way as above. Therefore, we conclude that assertion (40) is true.

Finally, it remains to show that $\eta^\infty = 0$. It follows from relations (30) and (40) that the right-hand side of Eq. (25) tends to $-(\eta^\infty)^2 H_0(\eta^\infty)$ as $\phi \to +\infty$. Furthermore, we have that

$$\lambda_*^2 (Q_\lambda)(\phi) \to \lambda_*^2 \eta^\infty \int_{-\infty}^{+\infty} q_\lambda(\varphi)\, \mathrm{d}\varphi = (\lambda - 1)\eta^\infty \quad \text{as } \phi \to +\infty.$$

If for some sequence $\{\phi_k\}_1^\infty$ tending to $+\infty$ we have:

$$\left[(\mathrm{d}_\phi Q_\lambda \mathrm{d}_\varphi)\eta\right](\phi_k) \to 0 \quad \text{as } k \to \infty, \tag{43}$$

then we obtain the following equation for η^∞:

$$(1 - \lambda)\eta^\infty = \left(\eta^\infty\right)^2 H_0\left(\eta^\infty\right).$$

Using the first formula (21), one finds after some algebra that this equation has two roots greater than -1 (the bottom level). They are equal to 0 and $\eta_* \leqslant 0$, and so $\eta^\infty = 0$ because $\eta(\phi)$ is assumed to be non-negative for large values of ϕ.

Let us demonstrate that relation (43) holds for some $\phi_k \to +\infty$, $k \to \infty$. The first step is to show that for any $t \in \mathbb{R}$ we have:

$$\left\|(\mathrm{d}_\phi Q_\lambda \mathrm{d}_\varphi)\eta\right\|_{L^2(t,t+1)} \leqslant C \int_{-\infty}^{+\infty} \mathrm{e}^{-\pi|t-\tau|} \|\eta_\varphi\|_{L^2(\tau,\tau+1)} \mathrm{d}\tau. \tag{44}$$

Let us split $(\mathrm{d}_\phi Q_\lambda \mathrm{d}_\varphi)\eta$ into a sum:

$$(\mathrm{d}_\phi Q_\lambda)(\eta_\varphi \chi_{[t-1,t+2]}) + (\mathrm{d}_\phi Q_\lambda)\left[(1 - \chi_{[t-1,t+2]})\eta_\varphi\right] = w_1 + w_2,$$

where $\chi_{[t-1,t+2]}$ is the indicator function of $[t-1, t+2]$. Since $\mathrm{d}_\phi Q_\lambda$ is a singular integral operator, the following estimates are true:

$$\|w_1\|_{L^2(t,t+1)} \leqslant \|w_1\|_{L^2(\mathbb{R})} \leqslant C\|\eta_\varphi\|_{L^2(t-1,t+2)}. \tag{45}$$

On the other hand, using the second inequality (28), we obtain:

$$\|w_2\|_{L^2(t,t+1)} \leqslant C \int_{-\infty}^{+\infty} \mathrm{e}^{-\pi|t-\varphi|} |\eta_\varphi|\, \mathrm{d}\varphi. \tag{46}$$

Now inequality (44) follows from (45) and (46).

It remains to note that one obtains (43) from the fact that

$$\left\|(\mathrm{d}_\phi Q_\lambda \mathrm{d}_\varphi)\eta\right\|_{L^2(t,t+1)} \to 0 \quad \text{as } t \to +\infty,$$

which, in its turn, is a consequence of inequality (44) and relation (40). The proof is complete.

4. Theorems' proofs

4.1. Proof of Theorem 1

In order to prove Theorem 1, it is sufficient to establish that η vanishes identically. Indeed, in this case problem (2)–(5) takes the following form:

$$\phi_{xx} + \phi_{yy} = 0, \quad (x, y) \in \mathbb{R} \times (-1, 0);$$
$$\phi_y(x, -1) = \phi_y(x, 0) = 0; \qquad \left|\phi_x(x, 0)\right| = 1,$$

where $x \in \mathbb{R}$ in the boundary conditions. The uniqueness theorem for the Cauchy problem for the Laplace equation guarantees that this problem has only one solution that satisfies condition (6), and this solution is $\phi = x + b$, where b is an arbitrary constant.

The fact that η vanishes identically we prove by contradiction. Thus we assume, contrary to the assertions stated, that for $\lambda \geqslant 1$ there exists a solution of problem $\mathrm{P}_{(\phi,\eta)}$ with non-zero η. Moreover, we suppose that this solution satisfies condition (8) when $\lambda > 1$.

Let us show that

$$\eta(x) > 0 \quad \text{for all } x \in \mathbb{R}. \tag{47}$$

For this purpose we need some results from our previous paper [13]. In particular, the following inequalities were proved in Theorem 2.3(i):

$$\eta_* \leqslant \inf_{x\in\mathbb{R}} \eta(x) \leqslant 0 \leqslant \sup_{x\in\mathbb{R}} \eta(x). \tag{48}$$

Moreover, Theorem 2.3(ii) implies that the second inequality (48) must be strict when there exists $x_- \in \mathbb{R}$ such that $\eta(x_-) = \inf_{x\in\mathbb{R}} \eta(x)$. Since $\eta_* = 0$ for $\lambda = 1$, no such x_- exists which yields formula (47) in this case.

If $\lambda > 1$, then the above consideration leads to a contradiction with our assumption (8). Thus, relation (47) is proved for $\lambda \geqslant 1$. Now (47) and (48) imply that $\inf_{x\in\mathbb{R}} \eta(x) = 0$.

Using remark made after formula (18), we write $\eta(\phi)$ instead of $\eta(x(\phi, 1))$ in the remaining part of the proof. Without loss of generality we can assume that $\liminf_{\phi\to+\infty} \eta(\phi) = 0$. Then applying lemma, we obtain:

$$\eta(\phi) \to 0 \quad \text{as } \phi \to +\infty. \tag{49}$$

Our next aim is to define an appropriate linear function v_ρ. For this purpose we consider the bundle of straight lines that have negative slope in the positive ϕ-direction and go through the point $(0, \rho)$, where $\rho \in (0, \eta(0))$. (In fact, only sufficiently small values of ρ will be involved, at least such that $\rho < \eta(\phi)$ for $\phi \in [0, 1]$.) If the absolute value of slope is large, then the corresponding line has no common points with the graph of $\eta(\phi)$ in the first quadrant. Since (49) holds for η, any line, whose negative slope is sufficiently close to zero, does intersect the graph of η in the first quadrant. Hence for a given value of ρ the bundle contains the straight line with the largest absolute value of slope that is tangent to the graph of η in the first quadrant. The property of ϕ established in Lewy's theorem guarantees that the hodograph transform preserves the property of η to be real-analytic. Therefore, there is no interval, where the latter straight line is tangent to the graph of η. Consequently, we can denote by ϕ_+ (our notation omits the dependence of this quantity on ρ) the projection of the rightmost tangency point (if there are several such points corresponding to the same value of ρ) onto the ϕ-axis. Moreover, relation (49) implies the following two facts:

$$\phi_+ \quad \text{is non-decreasing when } \rho \text{ decreases};$$
$$\phi_+ \to +\infty, \quad \text{whereas } \eta_\phi(\phi_+) \to 0 \text{ as } \rho \to 0. \tag{50}$$

Finally, we put:

$$v_\rho(\varphi) = \eta_\phi(\phi_+)\,(\varphi - \phi_+) + \eta(\phi_+), \quad \varphi \in \mathbb{R},$$

where, by the definition, $\eta_\phi(\phi_+) < 0$.

Let us consider the following expression:

$$\big[(\mathrm{d}_\phi Q_\lambda \mathrm{d}_\varphi)\eta\big](\phi_+) = \int_{-\infty}^{+\infty} q_\lambda'(\phi_+ - \varphi)\eta_\varphi(\varphi)\,\mathrm{d}\varphi,$$

that appears in the left-hand side of Eq. (25). We write this expression as follows:

$$\begin{aligned}&\int_{-\infty}^{+\infty} q_\lambda'(\phi_+ - \varphi)\big[\eta(\varphi) - v_\rho(\varphi) + v_\rho(\varphi)\big]'\,\mathrm{d}\varphi\\ &\quad= \int_{-\infty}^{+\infty} q_\lambda''(\phi_+ - \varphi)\big[\eta(\varphi) - v_\rho(\varphi)\big]\,\mathrm{d}\varphi + \int_{-\infty}^{+\infty} q_\lambda(\phi_+ - \varphi)v_\rho''(\varphi)\,\mathrm{d}\varphi,\end{aligned}$$

where the last integral vanishes, whereas the first integral in the right-hand side is well-defined. Indeed,

$$q_\lambda''(\phi_+ - \varphi) = O\big(|\phi_+ - \varphi|^{-2}\big),$$

and the second factor of the integrand has a double zero at ϕ_+. Therefore, at this point Eq. (25) takes the form:

$$\begin{aligned}&\int_{-\infty}^{+\infty} q_\lambda''(\phi_+ - \varphi)\big[\eta(\varphi) - v_\rho(\varphi)\big]\,\mathrm{d}\varphi + \lambda_*^2 \int_{-\infty}^{+\infty} q_\lambda(\phi_+ - \varphi)\eta(\varphi)\,\mathrm{d}\varphi\\ &\quad= \eta_\phi^2(\phi_+)H_1\big(\eta(\phi_+), \eta_\phi(\phi_+)\big) - \eta^2(\phi_+)H_0\big(\eta(\phi_+)\big).\end{aligned} \tag{51}$$

Let us estimate the first term in the left-hand side, for which purpose we split it into a sum:

$$\int_{-\infty}^{+\infty} q_\lambda''(\phi_+ - \varphi)\big[\eta(\varphi) - v_\rho(\varphi)\big]\,\mathrm{d}\varphi = \left(\int_{-\infty}^{\phi_-} + \int_{\phi_-}^{+\infty}\right) q_\lambda''(\phi_+ - \varphi)\big[\eta(\varphi) - v_\rho(\varphi)\big]\,\mathrm{d}\varphi, \tag{52}$$

where ϕ_- is the ϕ-coordinate of the rightmost point, at which the graphs of v_ρ and η do intersect in the second quadrant. (Note that $\phi_- \to -\infty$ as $\rho \to 0$.) Therefore, the second integral in the right-hand side of (52) is non-negative, whereas the first one is greater than or equal to

$$-\int_{-\infty}^{\phi_-} q_\lambda''(\phi_+ - \varphi)v_\rho(\varphi)\,\mathrm{d}\varphi = v_\rho(\phi_-)q_\lambda'(\phi_+ - \phi_-) + \eta_\phi(\phi_+)q_\lambda(\phi_+ - \phi_-). \tag{53}$$

Here the last expression is obtained by virtue of integration by parts. Thus we arrive at the following consequence of Eq. (51):

$$\begin{aligned}&\int_{\phi_-}^{+\infty} q_\lambda''(\phi_+ - \varphi)\big[\eta(\varphi) - v_\rho(\varphi)\big]\,\mathrm{d}\varphi + \lambda_*^2 \int_{-\infty}^{+\infty} q_\lambda(\phi_+ - \varphi)\eta(\varphi)\,\mathrm{d}\varphi\\ &\quad\leqslant \eta_\phi^2(\phi_+)H_1\big(\eta(\phi_+), \eta_\phi(\phi_+)\big) - \eta^2(\phi_+)H_0\big(\eta(\phi_+)\big) + C_1 \mathrm{e}^{-\pi(\phi_+ - \phi_-)}\big[v_\rho(\phi_-) - \eta_\phi(\phi_+)\big],\end{aligned} \tag{54}$$

where the first two inequalities (28) are applied for estimating the right-hand side in (53).

Let $\phi_1 \in (\phi_+, +\infty)$ be such that $v_\rho(\phi_1) = 0$. Since both integrands in the left-hand side of (54) are non-negative, this left-hand side is estimated from below by:

$$\int_{\phi_-}^{+\infty} q_\lambda''(\phi_+ - \varphi)\big[\eta(\varphi) - v_\rho(\varphi)\big]\,\mathrm{d}\varphi \geqslant (\eta_1 - \rho)\int_0^1 q_\lambda''(\phi_+ - \varphi)\,\mathrm{d}\varphi - \int_{\phi_1}^{+\infty} q_\lambda''(\phi_+ - \varphi)v_\rho(\varphi)\,\mathrm{d}\varphi, \tag{55}$$

where $\eta_1 = \min_{\phi\in[0,1]} \eta(\phi)$ and is assumed to be grater than ρ. The right-hand side in (55) is equal to

$$(\eta_1 - \rho)\big[q'_\lambda(\phi_+) - q'_\lambda(\phi_+ - 1)\big] - \eta_\phi(\phi_+)\, q_\lambda(\phi_1 - \phi_+), \tag{56}$$

where the last term is obtained integrating by parts in the last integral in (55). Since formula (26) yields that

$$q'_\lambda(\phi) = \frac{-\mathrm{e}^{-\pi\phi}}{1 + (\lambda_*/\pi)^2} + O\big(\mathrm{e}^{-2\pi\phi}\big) \quad \text{as } \phi \to +\infty,$$

the first term in (56) has the following behaviour:

$$\mathrm{e}^{-\pi\phi_+}\left[\frac{\pi^2(\mathrm{e}^\pi - 1)(\eta_1 - \rho)}{\pi^2 + \lambda_*^2} + O\big(\mathrm{e}^{-\pi\phi_+}\big)\right],$$

when ρ is sufficiently small, and so ϕ_+ is large. Hence for such values of ρ we get from (54) that

$$\begin{aligned} &-\eta_\phi(\phi_+)q_\lambda(\phi_1 - \phi_+) + C(\lambda_*, \eta_1)\mathrm{e}^{-\pi\phi_+} \\ &\quad \leqslant \eta_\phi^2(\phi_+)H_1\big(\eta(\phi_+), \eta_\phi(\phi_+)\big) - \eta^2(\phi_+)H_0\big(\eta(\phi_+)\big) + C_1\mathrm{e}^{-\pi(\phi_+ - \phi_-)}\big[v_\rho(\phi_-) - \eta_\phi(\phi_+)\big], \end{aligned} \tag{57}$$

where

$$C(\lambda_*, \eta_1) = \frac{\pi^2(\mathrm{e}^\pi - 1)(\eta_1 - \rho)}{2(\pi^2 + \lambda_*^2)}.$$

Investigating inequality (57), one has to consider two cases that depend on whether $\phi_1 - \phi_+$ is less than one or not. First we assume that $\phi_1 - \phi_+ < 1$, in which case there exists $M > 0$ such that $q_\lambda(\phi_1 - \phi_+) \geqslant M$ (see formula (29)). Then we get from (57) the following inequality:

$$\begin{aligned} C(\lambda_*, \eta_1)\mathrm{e}^{-\pi\phi_+} &\leqslant \eta_\phi(\phi_+)\big[M + \eta_\phi(\phi_+)H_1\big(\eta(\phi_+), \eta_\phi(\phi_+)\big) - C_1\mathrm{e}^{-\pi(\phi_+ - \phi_-)}\big] \\ &\quad - \eta^2(\phi_+)H_0\big(\eta(\phi_+)\big) + C_1\mathrm{e}^{-\pi(\phi_+ - \phi_-)}v_\rho(\phi_-). \end{aligned} \tag{58}$$

The expression in the square brackets is positive, since (50) implies that the terms depending on ϕ_+ tend to zero as $\phi_+ \to +\infty$ (that is, when $\rho \to 0$). Taking into account that $\eta_\phi(\phi_+) < 0$, we obtain from inequality (58) that

$$C(\lambda_*, \eta_1) \leqslant C_1\mathrm{e}^{\pi\phi_-}v_\rho(\phi_-) = C_1\mathrm{e}^{\pi\phi_-}\eta(\phi_-).$$

Note that the right-hand side goes to zero as $\rho \to 0$, and so $\phi_- \to -\infty$. Hence the assumption that $\phi_1 - \phi_+ < 1$ leads to a contradiction and it remains to consider the case when $\phi_1 - \phi_+ \geqslant 1$.

When $\phi_1 - \phi_+ \geqslant 1$ the definition of v_ρ implies that $|\eta_\phi(\phi_+)| \leqslant \eta(\phi_+)$. Moreover, in view of the behaviour of $\eta(\phi_+)$ and $\eta_\phi(\phi_+)$ formulae (21) give us that

$$H_0\big(\eta(\phi_+)\big) = \frac{3\lambda^2}{2} + o(1), \qquad H_1\big(\eta(\phi_+), \eta_\phi(\phi_+)\big) = \frac{1}{2} + o(1) \quad \text{as } \phi_+ \to +\infty$$

(that is, when $\rho \to 0$), and so we have that

$$\eta_\phi^2(\phi_+)H_1\big(\eta(\phi_+), \eta_\phi(\phi_+)\big) - \eta^2(\phi_+)H_0\big(\eta(\phi_+)\big) \leqslant 0$$

when ϕ_+ is sufficiently large (that is, ρ is close to zero) Therefore, inequality (57) and the fact that the first term in the right-hand side of that inequality is positive imply that

$$C(\lambda_*, \eta_1) \leqslant C_1\mathrm{e}^{\pi\phi_-}\big[v_\rho(\phi_-) - \eta_\phi(\phi_+)\big].$$

Noting that $v_\rho(\phi_-) = \eta(\phi_-)$, the right-hand side tends to zero as $\rho \to 0$, and so $\phi_- \to -\infty$. The proof is complete.

4.2. Proof of Theorem 2

Let us assume, contrary to assertion (I), that η is non-negative or non-positive for the values of ϕ that are greater than a certain number. In the first case, we apply lemma and get that $\eta(\phi) \to 0$ as $\phi \to +\infty$. If η is non-positive for large values of ϕ, then Theorem 4.3 in [13] implies that η tends to either 0 or η_* at the positive (or/and negative) infinity. Thus assertion (II) holds when (I) does not.

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